

# THE PYTHAGOREAN THEOREM IN CERTAIN SYMMETRY CLASSES OF TENSORS

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The Hadamard determinant theorem [1] states that if  $A = (a_{ij})$  is a positive semi-definite  $k$ -square Hermitian matrix then

$$(1) \quad \det A \leq \prod_{i=1}^k a_{ii}$$

with equality if and only if  $A$  has a zero row or  $A$  is diagonal.

In a recent paper [3] it was conjectured that an analogous result to (1) holds for the permanent of  $A$ . We recall that the permanent is defined by

$$\text{per}(A) = \sum_{\sigma \in S_k} \prod_{i=1}^k a_{\sigma(i)i}$$

where the summation is over the whole symmetric group of degree  $k$ . Recent interest in the permanent function stems from its application to a variety of combinatorial problems [4] and a partly unresolved conjecture of B. L. van der Waerden [2]. In [3] it was suggested that if  $A$  is once again a positive semi-definite  $k$ -square Hermitian matrix then

$$(2) \quad \text{per}(A) \geq \prod_{i=1}^k a_{ii}$$

with equality if and only if  $A$  has a zero row or  $A$  is diagonal. We are as yet unable to prove this but the subsequent inequality (3) is a step in this direction.

The first purpose of this note is to exhibit (1) as a case of the Pythagorean Theorem in a suitable symmetry class of tensors. Of course, many proofs of (1) are extant and our purpose in reproving it here is to exhibit a technique that is proving itself useful for examining a wide variety of matrix functions. We then show by a similar approach that

$$(3) \quad \text{per}(A) \geq \left( \prod_{i=1}^k a_{ii} \right) k! / k^{2k}$$

where the inequality is strict unless  $A$  has a zero row.

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The technique here is to regard the determinant and permanent functions as analytic expressions for inner products in suitable symmetry classes of tensors. To be explicit, let  $U$  be an  $n$ -dimensional unitary vector space with an inner product  $(x, y)$ . Let  $U^{(k)}$  be the space of  $k$ -tensors over  $U$ ; i.e., the  $n^k$ -dimensional dual space of the vector space  $M^{(k)}$  of multilinear functionals  $\phi$  of  $k$ -tuples of vectors from  $U$ . Certain distinguished "pure" vectors in  $U^{(k)}$  are denoted by  $f = x_1 \otimes \cdots \otimes x_k$  where  $x_i \in U$  and  $f$  is defined by  $f(\phi) = \phi(x_1, \cdots, x_k)$  for each  $\phi \in M^{(k)}$ . The pure vectors span  $U^{(k)}$  and the conjugate bilinear functional defined on pure vectors by

$$(4) \quad (x_1 \otimes \cdots \otimes x_k, y_1 \otimes \cdots \otimes y_k) = \prod_{i=1}^k (x_i, y_i)$$

is extendable to a unitary inner product on  $U^{(k)}$ . Let  $T$  and  $S$  be the symmetry operators of  $U^{(k)}$  into itself defined by

$$(5) \quad T(x_1 \otimes \cdots \otimes x_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \epsilon(\sigma) x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)},$$

$$(6) \quad S(x_1 \otimes \cdots \otimes x_k) = \frac{1}{k!} \sum_{\sigma \in S_k} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)},$$

where  $\epsilon(\sigma) = \pm 1$  according as  $\sigma$  is even or odd.

It is well known that  $T$  and  $S$  are Hermitian (with respect to the inner product in (4)) and idempotent. Moreover, one computes easily that

$$(Tx_1 \otimes \cdots \otimes x_k, y_1 \otimes \cdots \otimes y_k) = \frac{1}{k!} \det((x_i, y_j))$$

and

$$(Sx_1 \otimes \cdots \otimes x_k, y_1 \otimes \cdots \otimes y_k) = \frac{1}{k!} \text{per}((x_i, y_j)).$$

To prove (1) we first remark that if  $A$  is singular,  $\det A = 0$ , then the inequality obviously holds and equality requires that some  $a_{ii} = 0$ . But then  $0 \leq a_{ii}a_{jj} - |a_{ij}|^2 = -|a_{ij}|^2$  and hence  $a_{ij} = 0$ ,  $j = 1, \cdots, k$ , and row  $i$  of  $A$  is zero. Hence assume  $A$  is nonsingular and let  $x_1, \cdots, x_k$  be a set of linearly independent vectors such that  $(x_i, x_j) = a_{ij}$ ,  $i, j = 1, \cdots, k$ . Let  $u_1, \cdots, u_k$  be the E. Schmidt orthonormalizing sequence for  $x_1, \cdots, x_k$ . That is, the space  $\langle x_1, \cdots, x_p \rangle$  spanned by  $x_1, \cdots, x_p$  is the same as the space  $\langle u_1, \cdots, u_p \rangle$  spanned by  $u_1, \cdots, u_p$ ,  $p = 1, \cdots, k$ . Then since  $u_{i_1} \otimes \cdots \otimes u_{i_k}$ ,  $1 \leq i_\alpha \leq k$ ,  $\alpha = 1, \cdots, k$ , is an orthonormal basis in  $U^{(k)}$  we have from the Pythagorean theorem that

$$\begin{aligned}
\frac{1}{k!} \det A &= \frac{1}{k!} \det((x_i, x_j)) = (Tx_1 \otimes \cdots \otimes x_k, Tx_1 \otimes \cdots \otimes x_k) \\
&= \sum | (Tx_1 \otimes \cdots \otimes x_k, u_{i_1} \otimes \cdots \otimes u_{i_k}) |^2 \\
&= \left( \frac{1}{k!} \right)^2 \sum | \det((x_s, u_{i_t})) |^2, \quad s, t = 1, \cdots, k,
\end{aligned}$$

where the summation extends over all  $k^k$  ordered selections  $(i_1, \cdots, i_k)$  from  $1, \cdots, k$ . Since the determinant vanishes when two columns are the same, the last summation may be taken over sets of distinct ordered choices  $(i_1, \cdots, i_k)$ , i.e., over all  $k!$  permutations of  $1, \cdots, k$ . Hence

$$\begin{aligned}
\frac{1}{k!} \det A &= \left( \frac{1}{k!} \right)^2 \sum_{\sigma \in S_k} | \det((x_s, u_{\sigma(t)})) |^2 \\
&= \frac{1}{k!} | \det((x_s, u_t)) |^2 \\
&= \frac{1}{k!} \left| \det \begin{bmatrix} (x_1, u_1) & 0 & \cdots & 0 \\ (x_2, u_1) & (x_2, u_2) & 0 & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ (x_k, u_1) & (x_k, u_2) & \cdots & (x_k, u_k) \end{bmatrix} \right|^2 \\
&= \frac{1}{k!} \prod_{i=1}^k | (x_i, u_i) |^2 \\
&\leq \frac{1}{k!} \prod_{i=1}^k (x_i, x_i) = \frac{1}{k!} \prod_{i=1}^k a_{ii}.
\end{aligned}$$

Now the equality holds by Schwarz's inequality if and only if  $x_\alpha$  is a multiple of  $u_\alpha$ . But since  $u_1, \cdots, u_k$  is an orthonormal set, it follows that  $A = ((x_i, x_j))$  is diagonal.

From (6) we compute that

$$\begin{aligned}
\text{per } A &= k!(Sx_1 \otimes \cdots \otimes x_k, x_1 \otimes \cdots \otimes x_k) \\
&= k! \| Sx_1 \otimes \cdots \otimes x_k \|^2 \\
&\geq k! | (Sx_1 \otimes \cdots \otimes x_k, u \otimes \cdots \otimes u) |^2 \\
(7) \quad &= k! | (x_1 \otimes \cdots \otimes x_k, Su \otimes \cdots \otimes u) |^2 \\
&= k! | (x_1 \otimes \cdots \otimes x_k, u \otimes \cdots \otimes u) |^2 \\
&= k! \prod_{i=1}^k | (x_i, u) |^2,
\end{aligned}$$

where  $u$  is any unit vector in  $U$ .

The problem then is to find for fixed  $x_i$ ,  $i = 1, \dots, k$ , a significant lower bound for the expression  $\prod_{i=1}^k |(x_i, u)|^2$  as a function of the unit vector  $u$ . We have the

**LEMMA.** *If  $x_1, \dots, x_k$  are vectors then there exists a unit vector  $u$  such that*

$$|(x_s, u)| \geq \|x_s\|/k, \quad s = 1, \dots, k.$$

**Proof.** Let  $y_i = x_i/\|x_i\|$  or  $-x_i/\|x_i\|$  so that  $\sum_{i=1}^k y_i$  is of maximal length. That is,  $\|\sum_{i=1}^k y_i\| \geq \|\sum_{i=1}^k \pm x_i/\|x_i\|\|$  for all choices of signs on the  $x_i$ . We assert that

$$(8) \quad \operatorname{Re}\left(y_s, \sum_{i=1, i \neq s}^k y_i\right) \geq 0, \quad s = 1, \dots, k.$$

For let  $z_s = \sum_{i=1, i \neq s}^k y_i$  and  $z = \sum_{i=1}^k y_i$  and we have

$$\|z\|^2 = \|y_s + z_s\|^2 \geq \|-y_s + z_s\|^2,$$

$$\|y_s\|^2 + 2 \operatorname{Re}(y_s, z_s) + \|z_s\|^2 \geq \|y_s\|^2 - 2 \operatorname{Re}(y_s, z_s) + \|z_s\|^2$$

and (8) follows.

It is clear that

$$(9) \quad \|z\| \leq \sum_{i=1}^k \|y_i\| = k$$

and thus

$$\begin{aligned} \left| \left( y_s, \frac{z}{\|z\|} \right) \right| &= \left| \left( y_s, \frac{y_s + z_s}{\|z\|} \right) \right| \\ (10) \quad &= \frac{1}{\|z\|} |(\|y_s\|^2 + (y_s, z_s))| \\ &\geq \frac{1}{\|z\|} \operatorname{Re}(1 + (y_s, z_s)) \geq \frac{1}{\|z\|} \geq \frac{1}{k}. \end{aligned}$$

In (7) we take  $u = z/\|z\|$  to obtain

$$\begin{aligned} \text{per } A &\geq k! \prod_{i=1}^k |(y_i, u)|^2 \|x_i\|^2 \\ &\geq (k!/k^{2k}) \prod_{i=1}^k \|x_i\|^2 \\ &= (k!/k^{2k}) \prod_{i=1}^k a_{ii}. \end{aligned}$$

Clearly if any  $x_j=0$  then  $\text{per } A=0$ ,  $a_{jj}=0$  and (3) is equality. If no  $x_j=0$  then (9) can be equality only if  $z=\theta ky_s$ , where  $|\theta|=1$ ,  $s=1, \dots, k$ , in which case

$$|(y_s, u)| = \left| \left( y_s, \frac{z}{\|z\|} \right) \right| = k/k = 1 > \frac{1}{k}$$

and hence the inequality in the lemma is strict for  $k>1$ .

Actually a refinement of the above argument shows that

$$\prod_{s=1}^k |(y_s, u)| \geq \frac{k^{1/2}}{k} \frac{k^{1/2}-1}{k-1} \cdots \frac{k^{1/2}-r}{k-r} \frac{1}{k^{k-r-1}},$$

where  $r = [k/(k^{1/2}+1)]$ , and therefore,

$$\text{per}(A) \geq \prod_{i=1}^k a_{ii} k! / \left( \frac{k^{1/2}}{k} \frac{k^{1/2}-1}{k-1} \cdots \frac{k^{1/2}-r}{k-r} \frac{1}{k^{k-r-1}} \right)^2.$$

For

$$(11) \quad \|z\|^2 = (z, z) = \sum_{s=1}^k (y_s, z) = k + \sum_{s=1}^k (y_s, z_s) = k + \sum_{s=1}^k \text{Re}(y_s, z_s) \geq k,$$

i.e.,  $\|z\| \geq k^{1/2}$ . We can assume without loss of generality that  $\text{Re}(y_1, z) \geq \cdots \geq \text{Re}(y_k, z)$ . Now from (11) we have

$$\|z\| = \sum_{s=1}^k \text{Re} \left( y_s, \frac{z}{\|z\|} \right) \geq k^{1/2}$$

and therefore  $\text{Re}(y_1, z/\|z\|) \geq k^{1/2}/k$ , and a fortiori,  $|(y_1, z/\|z\|)| \geq k^{1/2}/k$ . Further

$$\left| \sum_{s=2}^k \left( y_s, \frac{z}{\|z\|} \right) \right| \geq \left| \sum_{s=1}^k \left( y_s, \frac{z}{\|z\|} \right) \right| - \left| \left( y_1, \frac{z}{\|z\|} \right) \right| \geq k^{1/2} - 1$$

and thus  $|(y_2, z/\|z\|)| \geq (k^{1/2}-1)/(k-1)$ .

In general

$$(12) \quad \left| \left( y_{s+1}, \frac{z}{\|z\|} \right) \right| \geq \frac{k^{1/2}-s}{k-s}.$$

Comparing (12) with (10) we find, by an elementary computation, that

$$\max \left\{ \frac{k^{1/2}-s}{k-s}, \frac{1}{k} \right\} = \begin{cases} \frac{k^{1/2}-s}{k-s} & \text{for } s \leq r, \\ \frac{1}{k} & \text{for } s > r, \end{cases}$$

where  $r$  is the greatest integer in  $k/(k^{1/2}+1)$ .

Thus (12) gives a better lower bound for  $s=1, \dots, r$  while (10) gives a better lower bound for  $s=r+1, \dots, k$ . The result follows.

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